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Discrete systems related to the sixth Painlevé equation

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Abstract

We present discrete Painlevé equations which can be obtained as contiguity relations of the solutions of the continuous Painlevé VI. The derivation is based on the geometry of the affine Weyl group $D_4^{(1)}$ associated with the bilinear formalism. As an offshoot we also present the contiguity relations of the solutions of the Bureau–Ablowitz–Fokas equation, which is a Miura transformed, 'modified', P_{VI}.

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1. Introduction

Among all the interesting properties of the Painlevé equations the contiguity relations of their solutions play a very special role. They provide a link between the Painlevé equations and their discrete counterparts. A contiguity relation establishes an identity between the solutions of an equation (for the same value of the continuous variable) obtained for values of the parameters which differ by an integer multiple of an elementary step. Since no derivatives of the solution of the continuous equation appear in the contiguity relation the latter can be viewed as a discrete evolution equation in which the parameters of the continuous system play the role of the (discrete) independent variable (while the continuous variable is now relegated to the role of a mere parameter). One of the first occurrences of the discrete Painlevé equation in the literature was in the work of Jimbo and Miwa [1] who obtained the contiguity relation of the continuous P_{II} (but did not proceed to show that the resulting mapping was a discrete analogue of P_I). The construction of a contiguity relation of a continuous Painlevé equation follows a systematic procedure [2]. One starts from two auto-Bäcklund or Schlesinger transformations of a continuous Painlevé equation which relate the solution for three adjacent values of some parameter and eliminates all instances of the derivative of the solution. The resulting equation is a discrete Painlevé equation and the main advantage of this procedure is that it furnishes also the Lax pair of the discrete system [3].

The 'contiguity' approach has been successfully used for the construction of discrete Painlevé equations. In this paper we shall focus on the discrete systems obtained from the contiguity relations of the solutions of P_{VI}. Already in [4] we have studied some discrete systems which are related to the Painlevé VI equation through the similarity reductions of a two-dimensional lattice equation of KdV type. (These results were complemented and extended in [5] by Nijhoff, Joshi and Hone). Here we shall explore the contiguity relations of the solutions of P_{VI} in a systematic way with the help of a geometrical approach based on affine Weyl groups. The geometrical description of the transformations of the continuous Painlevé equations was introduced by Okamoto [6] and extended to the case of discrete Painlevé equations by the present authors in collaboration with Satsuma [7]. It has been cast by Sakai, in his ground-breaking work [8], into the proper perspective providing the complete classification of discrete Painlevé equations, which arise as reductions of the master system associated with $E_8^{(1)}$. Sakai obtained both difference and q-type equations and, moreover, discovered a third species, namely that of elliptic discrete Painlevé equations. In this paper we shall proceed from the explicit construction of the geometry of the weight lattice of the affine Weyl group $D_4^{(1)}$ to the derivation of the elementary Miura transformation. The contiguity relations are obtained by choosing a trajectory obtained by infinite repetitions of a non-closed pattern on the weight lattice of $D_4^{(1)}$ and combining appropriately the Miura transformations.

2. The Painlevé VI equation

The canonical form of the sixth Painlevé equation is

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}t^2} &= \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\frac{\mathrm{d}w}{\mathrm{d}t}^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{\mathrm{d}w}{\mathrm{d}t} \\ &\quad + \frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left(\theta_\infty^2 - \frac{\theta_0^2 t}{w^2} + \frac{\theta_1^2(t-1)}{(w-1)^2} + \frac{(1-\theta_t^2)t(t-1)}{(w-t)^2}\right). \qquad (2.1)\end{aligned}$$

In this form P_{VI} has solutions with fixed singularities at ∞, 0,1 and t, the latter being the independent variable. The θ_i are just the monodromy exponents at the four singularities. Equation (2.1) is invariant under the six-elements group generated by $\{x \to 1-x, t \to 1-t\}$, $\{x \to 1/x, t \to 1/t\}$.

However equation (2.1) is but one possible form of P_{VI}. A more general homographic change [9] of the dependent variable (involving three free functions) and a further free change of the independent variable can set the singularities at completely arbitrary positions $a(t), b(t), c(t), d(t)$. The resulting equation is rather awkward looking, but simplifies somewhat if one chooses as independent variable

$$T(t) = \frac{(d(t)-b(t))(c(t)-a(t))}{(d(t)-a(t))(c(t)-b(t))}$$

in which case it becomes

$$\begin{aligned}\ddot{w} &= \frac{1}{2}\left(\frac{1}{w-a}+\frac{1}{w-b}+\frac{1}{w-c}+\frac{1}{w-d}\right)\dot{w}^2 \\ &\quad - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{\dot{a}}{w-a}+\frac{\dot{b}}{w-b}+\frac{\dot{c}}{w-c}+\frac{\dot{d}}{w-d}+e\right)\dot{w} \\ &\quad + (w-a)(w-b)(w-c)(w-d) \\ &\quad \times \left(\frac{f}{(w-a)^2}+\frac{g}{(w-b)^2}+\frac{h}{(w-c)^2}+\frac{j}{(w-d)^2}\right), \qquad (2.2)\end{aligned}$$

where the dot indicates derivative with respect to T while e and f, g, h, j are lengthy expressions, the latter four being of the form $f = f_0\theta_a^2 + f_1$ and so on. From now on we will denote the independent variable as t and the derivative with respect to it by a prime.

When one is essentially interested in the continuous evolution along the independent variable, the use of the canonical form (2.1) is usually preferred. But when one is interested in the discrete equations which are nonlinear contiguities of solutions with different values of the monodromy exponents, the noncanonical form (2.2) leads to useful insights. A basic contiguous solution is provided by the following auto-Bäcklund transformation. If w satisfies (2.1) then the same equation for different values of the monodromy exponents Θ_i's ($i = \infty, 0, 1, t$) will be satisfied by W defined through [5, 6]

$$\frac{1-\sum\theta_i}{W-w} = \frac{t(t-1)w'}{w(w-1)(w-t)} + \frac{\theta_0}{w} + \frac{\theta_1}{w-1} + \frac{\theta_t - 1}{w-t} \tag{2.3}$$

and

$$\Theta_j = \theta_j + \frac{1}{2} - \frac{1}{2}\sum\theta_i = \left(\theta_j - \frac{1}{2}\right) - \frac{1}{2}\sum\left(\theta_i - \frac{1}{2}\right). \tag{2.4}$$

This elementary auto-Bäcklund transformation can be written as the product of two Miura transformations to another object ϕ which satisfies a different second-order equation. The first Miura transformation is

$$\phi = t\frac{w'}{w} + \frac{\theta_\infty}{2(t-1)}w + \frac{\theta_0}{2(t-1)}\frac{t}{w} - \frac{(\theta_\infty+\theta_0)(t+1)}{2(t-1)} - \frac{1}{2}, \tag{2.5a}$$

$$w = \frac{t(t^2-1)\phi + 2t(\kappa\phi+\mu) + t(t-1)\Omega}{2t(t-1)\phi' - (t-1)(\phi^2+\nu) - (t+1)(\kappa\phi+\mu)}, \tag{2.5b}$$

where

$$\Omega = \frac{2t(t-1)^2\phi'' + (t-1)(3t-1)\phi' + 4\phi(\phi^2+\nu) - 2\kappa(\kappa\phi+\mu)}{2(t+1)\phi + \lambda(t-1)} \tag{2.6}$$

and $\kappa = \theta_0 - \theta_\infty - 1$, $\lambda = \theta_0 + \theta_\infty$, $2\mu = \theta_1^2 - \theta_t^2$ and $2\nu = -\theta_1^2 - \theta_t^2 + \kappa^2/2$.

Eliminating ϕ in (2.5) leads to the P_{VI}, equation (2.1), for w, while eliminating w leads to an equation for ϕ:

$$t\Omega^2 = t(t-1)^2\phi'^2 + (\phi^2+\nu)^2 - (\kappa\phi+\mu)^2. \tag{2.7}$$

This equation has been described by Bureau in [10] as well as Ablowitz and Fokas in [11]. Note that it is of second degree in ϕ''.

A second Miura transformation exists in which θ_1 and θ_t would play the role of θ_∞ and θ_0 in (2.5). As a result, the auto-Bäcklund transformation (2.3) is the combination of these two 'orthogonal' Miuras.

3. Derivation of the contiguity relations

The best way to look at the Miura and auto-Bäcklund transformations of Painlevé equations is to express the solutions in terms of τ-functions which are entire functions of the independent variable t and depend on the monodromy exponents. For any *fixed* value of t, the τ-functions satisfy bilinear Hirota–Miwa equations of the form

$$H_1\bar{\tau}\underline{\tau} + H_2\tilde{\tau}\underset{\sim}{\tau} + H_3\hat{\tau}\underset{\wedge}{\tau} = 0, \tag{3.1}$$

where the accents $\bar{}$, $\tilde{}$ and $\hat{}$ over and under τs describe reciprocal modifications of the monodromy exponents. One of the advantages of this approach, introduced by Okamoto [6], is to classify these nonlinear contiguity relations which are in fact discrete Painlevé equations. Note that such a classification in terms of affine Weyl groups can be extended, as was shown by Sakai [8], to discrete Painlevé equations which are *not* contiguity relations of the solutions of (continuous) Painlevé equations. As Okamoto has noted, τ-functions 'live' on the vertices of the weight lattice of some affine Weyl group. In the case of discrete equations related to P_{VI} the relevant affine Weyl group is $D_4^{(1)}$.

This weight lattice can be described by two equivalent representations. One such representation of the weight lattice is the subset of $\mathbb{Z}^4$ such that the sum of coordinates is even. In this case, the nearest neighbours (NN) of the origin are at the distance $D = \sqrt{2}$. There are 24 such NN of the form

$$\begin{bmatrix} \pm 1 \\ \pm 1 \\ 0 \\ 0 \end{bmatrix}$$

with two zero coordinates and two coordinates of absolute value 1. They are manifestly equivalent. The next nearest neighbours (NNN) of the origin are at distance 2. There are again 24 of them, eight on the four axes, with the only nonzero coordinate of absolute value 2, and 16 more with all coordinates of absolute value 1, with four independent signs. They are also equivalent, although they do not look so. This is related to the fact that in four dimensions, the diagonal of a hypercube has just twice the length of its side, and thus a vertex is at the same distance of an adjoining vertex as to the centre of this hypercube. (This is unique to four dimensions, though in eight dimensions, there exists a similar situation. The diagonal of the hypercube is twice the length of the diagonal of an elementary square, and thus a vertex is at the same distance of a next neighbouring vertex as to the centre of this hypercube. The dimension eight is of particular interest since eight is the largest number of dimensions of affine Weyl groups related to second-order Painlevé equations.)

The other representation has vertices with coordinates either all even or all odd. This is equivalent to the first representation, through a rotation and a scaling by a factor $\sqrt{2}$. Note that in that case the 24 NN of the origin have exactly the coordinates of the *next nearest neighbours* of the origin in the previous case, while the next nearest neighbours of the origin have two zero coordinates and two coordinates of absolute value 2 (that is, exactly twice the coordinates of the NN of the origin in the other representation) and are now manifestly equivalent. This shows that, as we said earlier, *both* the 24 NN of the origin, on the one hand, and the 24 next nearest neighbours of the origin, on the other hand, are equivalent to each other.

In [4] we used the first representation, and we wrote the relevant bilinear Hirota–Miwa equations. But for the purpose of the present paper we will use the second one, and we will write bilinear Hirota–Miwa equations which are equivalent, through gauge transformations which, however, we will not give explicitly.

The solutions w of P_{VI} live at midpoint of τs in NNN relative position and have two even and two odd coordinates. For instance, there is such a solution, say w_{12}, at the point x_{12} at

$$\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

which is the midpoint of four pairs of τs in NNN relative position:

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right\}, \qquad \left\{\begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

and the two pairs

$$\left\{\begin{bmatrix}-1\\-1\\1\\\pm1\end{bmatrix},\begin{bmatrix}-1\\-1\\-1\\\mp1\end{bmatrix}\right\}.$$

The notation of indices and exponents for points like x and variables like τ or w indicates by how many units one has moved away from the origin in the respective directions. In this case an exponent ω^i means that we have moved by one step up in the i direction, ω^{ii} means two steps up in the i direction and so on. Conversely ω_i means one step down in the i direction etc. Since one can move simultaneously in all four directions, more complicated combinations of exponents/indices can exist.

At this point one can comment on the relation between the (rather abstract) coordinates in the four-dimensional $\mathrm{D}_4^{(1)}$ space and θ_j. First, we must remember that *a priori*, homographic invariance allows us to shift the singularities from ∞, 0, 1 and t to four arbitrary functions a, b, c and d of t, so the index of any θ is irrelevant, only the set of the four θ_j is meaningful. Also, from (2.1) (or (2.2)) each θ is only defined up to a sign. We are now in a position to compute θ_j at each point of the lattice. The coordinates m_i of τs and xs in the lattice are integers (with further restrictions on parity, as explained above), but the relevant values n_i (which will appear later in the paper) must be understood as globally shifted by an arbitrary set of numbers f_i (*fixed once and for all*) i.e., $n_i = m_i + f_i$, $i = 1, \ldots 4$.

At point x_{12} above, the directions leading to τs in NNN position are

$$\pm\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\quad \pm\begin{bmatrix}1\\-1\\0\\0\end{bmatrix},\quad \pm\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\quad \text{and}\quad \pm\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}.$$

The four θs, up to a sign, are half the projection of the vector of components n_i, which at point x_{12} is just

$$\begin{bmatrix}f_1-1\\f_2-1\\f_3\\f_4\end{bmatrix}$$

in these four directions, respectively $\pm(f_1+f_2-2)/2$, $\pm(f_1-f_2)/2$, $\pm(f_3+f_4)/2$ and $\pm(f_3-f_4)/2$. Note that at point x_{13} (or x_1^3, or x_{24}, or x_{11}^{24}, etc) the arrangement would be different, mixing directions 1 and 3 on the one hand, 2 and 4 on the other hand, and the third combination (1 with 4, 2 with 3) will appear at points like x_{14}, x_2^3, x_{2344}, x_{113}^{24444} and so on.

The τ-functions obey many equations, among which some involve the continuous variable. For a *fixed* given value of the continuous variable the τ-functions obey bilinear equations on the $\mathrm{D}_4^{(1)}$ lattice. These equations are highly overdetermined but of course compatible. One set of equations are 'around' the positions x of the w variables. For instance around x_{12} we have the equations

$$C\tau\tau_{1122} - S\tau_{11}\tau_{22} = \tau_{1234}\tau_{12}^{34} \tag{3.2a}$$

$$S\tau\tau_{1122} - C\tau_{11}\tau_{22} = \tau_{123}^{4}\tau_{124}^{3}, \tag{3.2b}$$

where C and S obey the hyperbolic functions rule $C^2 - S^2 = 1$, and are related to the continuous variable of $\mathrm{P_{VI}}$. Note however that one cannot make this relation explicit until the precise values of the functions a, b, c, d and T in terms of t are given. All equations of the form (3.2) with all permutations of all four directions, as well as translations by vectors of the lattice (any vector with all its components of the same parity), are simultaneously satisfied on the lattice. Note however that the equations are *not* invariant under change of the *orientation* of the axes, not even up to simple modifications. For instance, changing the orientation of the '4'-axis cannot just be compensated by the interchange of S and C because this does not conserve the hyperbolic relation $C^2 - S^2 = 1$. Such a change is possible but only up to a very complicated gauge transformations on the τ-functions. One sees that any two of the products of two τ-functions on one of the four pairs of NNN around x_{12} determine the two others as linear combinations. As a result the ratio of any two of these products can be written as a homographic function of the ratio of any two of them (not necessarily both different). Since, as we remarked above, the actual value of the solution w_{12} is only determined up to an arbitrary homographic transformation it follows that until extraneous information is added to fix that transformation, the ratio of any two of these products is as good a definition of w_{12} as any other one. In particular the canonical form (2.1) can be recovered with

$$w \equiv w_{12} = \frac{S}{C}\frac{\tau_{123}^{4}\tau_{124}^{3}}{\tau_{1234}\tau_{12}^{34}}, \qquad a = \infty, \qquad b = 0, \qquad c = 1, \qquad d = T = \left(\frac{S}{C}\right)^2. \tag{3.3}$$

Just as in the case of x_{12} above, the index and exponent notation for the nonlinear variables and τs indicate the point on the lattice where the variable lives i.e. by how many units one has moved away from the origin in the respective directions.

Another interesting set of bilinear equations can be written around points which are midpoints of τs in NN relative position. As we have argued above, all these points are actually equivalent, and were explicitly so in the parametrization of the lattice used in [4]. In the present parametrization, however, this is not the case, and they fall into two apparent classes: points of the form

$$\xi_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with all integer coordinates, one of a given parity and the three others having the opposite parity on the one hand, and points with all coordinates half-integer, on the other hand. The NN pair of τs around ξ_1 are τ at the origin and τ_{11} at

$$\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Exactly four pairs of τs at square distance $D^2 = 12$ exist around this point. Indeed the eight points

$$\begin{bmatrix} -1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

where $\sigma_i^2 = 1$ are at square-distance $D^2 = 3$ from ξ_1 and two by two symmetrical with respect to it. The value of the variable at ξ_1 (which is nothing but the variable $\phi \equiv \phi_1$ of the

Bureau–Ablowitz–Fokas equation) has four expressions leading to more bilinear equations:

$$\begin{aligned}\phi_1 &= \frac{\tau_{1234}\tau_1^{234}}{\tau_{11}\tau} - (n_2+n_3+n_4)\\ &= \frac{\tau_{134}^2\tau_{12}^{34}}{\tau_{11}\tau} - (n_2-n_3-n_4)\\ &= \frac{\tau_{124}^3\tau_{13}^{24}}{\tau_{11}\tau} - (-n_2+n_3-n_4)\\ &= \frac{\tau_{123}^4\tau_{14}^{23}}{\tau_{11}\tau} - (-n_2-n_3+n_4).\end{aligned} \tag{3.4}$$

Note that at ξ_1, though $m_1 = -1$, one has $m_2 = m_3 = m_4 = 0$, and thus all the n_i that appear in (3.4) are equal to the respective shifts f_i.

Equating any of these expression to the three others leads to three independent equations. Multiplying by $\tau\tau_{11}$ we obtain three bilinear equations. Moreover there are two independent ways to combine these three equations in order to eliminate the product $\tau\tau_{11}$, in favour of products of τs at square distance 12 to each other. For instance

$$(n_2+n_4)\tau_{134}^2\tau_{12}^{34} - (n_3+n_4)\tau_{124}^3\tau_{13}^{24} + (n_3-n_2)\tau_{1234}\tau_1^{234} = 0. \tag{3.5}$$

Note that by construction, the sum of the coefficients of the products vanishes for all the equations obtained in this way (because in the equations involving $\tau\tau_{11}$ the coefficients of the other products are 1 and -1 respectively, and then linear combinations are taken). This important property has been dubbed the 'Hirota property' [12]. This property is *not* gauge invariant and is thus a rather arbitrary property of the gauge one is working with rather than an essential property of the system under study. However the reason we are using the present parametrization of the space is precisely because the gauge with the Hirota property is very natural with this parametrization, while it would be extremely awkward in the parametrization of [4].

Starting from (3.5) we divide by $\tau_{1234}\tau_{13}^{24}\tau_{12}^{34}$ and multiply by τ_{123}^4 obtaining

$$(n_2+n_4)\frac{\tau_{134}^2\tau_{123}^4}{\tau_{1234}\tau_{13}^{24}} - (n_3+n_4)\frac{\tau_{124}^3\tau_{123}^4}{\tau_{1234}\tau_{12}^{34}} + (n_3-n_2)\frac{\tau_1^{234}\tau_{123}^4}{\tau_{12}^{34}\tau_{13}^{24}} = 0, \tag{3.6}$$

which can provisionally be written as

$$(n_2+n_4)W_{13} - (n_3+n_4)W_{12} + (n_3-n_2)W_1^4 = 0, \tag{3.7}$$

where W s are solutions of *some* form of equation (2.2), containing all the information concerning the site they are on, and are a homographic function of the actual values of the variables of the $\mathrm{P_{VI}}$ equation we will actually choose. In fact, it is convenient to choose $W = 1/w$ for all three variables of (3.7). An elementary calculation shows that it follows that

$$\frac{n_2+n_4}{w_{12}-w_1^4} + \frac{n_3-n_2}{w_{12}-w_{13}} = \frac{n_3+n_4}{w_{12}}. \tag{3.8}$$

Because of the Hirota property of the bilinear equation, the numerator of the rhs is the sum of the numerators of the two terms on the lhs A slightly different choice of $W = 1/(w-\mu)$ would have been possible (corresponding to a singularity at μ) but here we chose simply to put the singularity at zero.

Eliminating ϕ_1 and $\tau\tau_{11}$ in (3.4) in a different way we get the equation

$$(n_4-n_3)\tau_{1234}\tau_1^{234} - (n_2+n_4)\tau_{14}^{23}\tau_{123}^4 + (n_2+n_3)\tau_{124}^3\tau_{13}^{24} = 0, \tag{3.9}$$

which (dividing by $\tau_{123}^4\tau_{124}^3\tau_1^{234}$ and multiplying by τ_{12}^{34}) leads to

$$(n_4-n_3)w_{12} - (n_2+n_4)w_1^3 + (n_2+n_3)w_1^4 = 0, \tag{3.10}$$

where $w_1^3 = \tau_{12}^{34}\tau_{14}^{23}/\left(\tau_1^{234}\tau_{124}^3\right)$. Here we get

$$-\frac{(n_2+n_4)}{w_1^4 - w_{12}} + \frac{(n_2+n_3)}{w_1^3 - w_{12}} = 0. \tag{3.11}$$

Even though the bilinear equation (3.9) still has the Hirota property, the sum of the numerators of the two terms on the lhs is not matched on the lhs. In fact a term $(n_4 - n_3)/(w_{12} - \rho)$ is formally present, but the relevant singularity ρ is infinite.

Bilinear equations around points with all coordinates half-integer also exist. These points are mid-points of τs in NN positions and are equivalent to points like ξ_1 (though the equivalence in this case is not manifest). Around the point ζ at

$$\begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

one could define a Bureau–Ablowitz–Fokas variable, analogous to ϕ_1. We will not, however, write the analogue of full (3.4) system but only the analogue of (3.5), namely the bilinear equation

$$(n_3 - n_2)\tau_{234}^1\tau_{11} + (n_1 - n_3)\tau_{134}^2\tau_{22} + (n_2 - n_1)\tau_{124}^3\tau_{33} = 0, \tag{3.12}$$

which also has the Hirota property. Dividing by $\tau_{11}\tau_{22}\tau_{33}$ and multiplying by τ_{123}^4 we obtain

$$(n_3 - n_2)\frac{\tau_{234}^1\tau_{123}^4}{\tau_{22}\tau_{33}} + (n_1 - n_3)\frac{\tau_{134}^2\tau_{123}^4}{\tau_{11}\tau_{33}} + (n_2 - n_1)\frac{\tau_{124}^3\tau_{123}^4}{\tau_{11}\tau_{22}} = 0, \tag{3.13}$$

which again can provisionally be written as

$$(n_3 - n_2)V_{23} + (n_1 - n_3)V_{13} + (n_2 - n_1)V_{12} = 0, \tag{3.14}$$

where V s contain the information of the solution of $\mathrm{P_{VI}}$ but maybe up to some homography. Careful calculations (using (3.2)) show that in fact $1/V_{12} = Sw_{12} - C$, and since everything is invariant upon permutation of the directions, $1/V_{13} = Sw_{13} - C$. Defining x_{23} in the same way in terms of V_{23}, we can write

$$\frac{n_3 - n_2}{w_{23} - \nu} + \frac{n_1 - n_3}{w_{13} - \nu} + \frac{n_2 - n_1}{w_{12} - \nu} = 0, \tag{3.15}$$

where now the singularity is at $\nu = C/S$. This equation can be rewritten in several ways, leading to different interpretations. For instance one can write:

$$\frac{n_3 - n_2}{w_{13} - w_{12}} + \frac{n_2 - n_1}{w_{13} - w_{23}} = \frac{n_3 - n_1}{w_{13} - \nu}. \tag{3.16}$$

Comparing with (3.8) one can give the following interpretation: we go from $w_{m-2} \equiv w_{23}$ to $w_{m-1} \equiv w_{13}$ to $w_m \equiv w_{12}$ to $w_{m+1} \equiv w_1^4$, and (3.16) and (3.8) are the $(m-1)$ and m instances of the formal expression:

$$\frac{Z_{m+1/2}}{w_m - w_{m+1}} + \frac{Z_{m-1/2}}{w_m - w_{m-1}} = \frac{Z_{m+1/2} + Z_{m-1/2}}{w_m - a_m}. \tag{3.17}$$

The points x_{13}, x_{12} and x_1^4 (and also x_{23}, x_{13} and x_{12}), where the relevant w live, form an equilateral triangle. The evolution corresponds to some complicated, spiral, trajectory where two successive triangles have a common side. In particular, we have identified one possible trajectory where on the rhs of (3.17) the variable a_m has period four taking successively the value of the four singularities, 0 (see (3.8)), $\nu = C/S$ (see (3.16)), and also $\sigma = S/C$ and $\rho \equiv \infty$. In the latter case the rhs will in fact vanish, thus giving the impression that the sum of the numerators of the lhs is not balanced by that of the rhs while in fact one must think in terms of a balancing numerator with an infinite denominator. Indeed, this precise denominator would appear if we were to apply a homographic transformation on w to bring the infinite singularity at finite distance. The Z_m are obtained as linear combinations of the n_i, and for the particular trajectory we are considering, having periodicity six in addition to a linear dependence on m: $Z_{m+1/2} = pm + q + k(-1)^m + rj^m + sj^{2m}$ where j is a cube root of unity.

A different contiguity relation is obtained if we reorganize the terms in (3.15) as

$$\frac{n_3 - n_2}{w_{12} - w_{13}} + \frac{n_1 - n_3}{w_{12} - w_{23}} = \frac{n_1 - n_2}{w_{12} - \nu}, \tag{3.18}$$

and subtract (3.18) from (3.8) to eliminate w_{13}. We obtain then

$$\frac{n_2 + n_4}{w_{12} - w_1^4} + \frac{n_3 - n_1}{w_{12} - w_{23}} = \frac{n_3 + n_4}{w_{12}} + \frac{n_2 - n_1}{w_{12} - \nu}. \tag{3.19}$$

By construction the sum of the numerators of the lhs is balanced by the sum of the numerators of the rhs. If we now define $w_{m-1} \equiv w_{23}$, $w_m \equiv w_{12}$ to $w_{m+1} \equiv w_1^4$, we formally get

$$\frac{Z_{m+1/2}}{w_m - w_{m+1}} + \frac{Z_{m-1/2}}{w_m - w_{m-1}} = \frac{Z_a}{w_m - a_m} + \frac{Z_b}{w_m - b_m}. \tag{3.20}$$

The points x_{23}, x_{12} and x_1^4, where the relevant w live, form a right isosceles triangle. Within the plane of the triangle we can define an evolution which follows a staircase trajectory. Equation (3.19) is then complemented by

$$\frac{n_3 - n_1}{w_{23} - w_{12}} + \frac{n_4 + n_2 - 2}{w_{23} - w_{2234}} = \frac{n_3 + n_2 - 2}{w_{23} - \sigma} + \frac{n_4 - n_1}{w_{23} - \rho}. \tag{3.21}$$

If we now define $w_{m-2} \equiv w_{2234}$, we obtain the $(m-1)$ occurrence of (3.20). Along this trajectory a_m alternates between the values 0 and $\rho \equiv \infty$ while b_m alternates between ν and σ. The corresponding Z_a and Z_b are $pm + q + r\,\mathrm{i}^m + s(-\mathrm{i})^m$ and $pm + q - r\,\mathrm{i}^m - s(-\mathrm{i})^m$, respectively, while we have $Z_{m+1/2} = p(m+1/2) + q + k(-1)^m$. Note that we have $Z_a + Z_b = Z_{m-1/2} + Z_{m+1/2}$.

Another possibility of contiguity can be obtained from

$$\frac{n_1 - n_3}{w_{12} - w_{23}} + \frac{n_4 + n_2 - 2}{w_{12} - w_{1224}} = \frac{n_1 + n_2 - 2}{w_{12} - \sigma} + \frac{n_4 - n_3}{w_{12} - \rho}, \tag{3.22}$$

where we have already seen that $\sigma = s/c$ and $\rho = \infty$. So the last term on the rhs of (3.22) is effectively absent. Adding (3.22) to (3.19) we eliminate w_{23} and get

$$\frac{n_2 + n_4}{w_{12} - w_1^4} + \frac{n_4 + n_2 - 2}{w_{12} - w_{1224}} = \frac{n_3 + n_4}{w_{12}} + \frac{n_2 - n_1}{w_{12} - \nu} + \frac{n_1 + n_2 - 2}{w_{12} - \sigma} \left\{ + \frac{n_4 - n_3}{w_{12} - \infty} \right\} \tag{3.23}$$

with a 'phantom' fourth term. The points x_{1224}, x_{12} and x_1^4, where the relevant w live, are on a straight line along the vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

With the notation $w_{m-1} \equiv w_{1224}$, $w_m \equiv w_{12}$ to $w_{m+1} \equiv w_1^4$, we formally get

$$\frac{Z_m + Z_{m+1}}{w_m - w_{m+1}} + \frac{Z_m + Z_{m-1}}{w_m - w_{m-1}} = \frac{Z_a}{w_m - a} + \frac{Z_b}{w_m - b} + \frac{Z_c}{w_m - c} + \frac{Z_d}{w_m - d}. \tag{3.24}$$

Here Z_m is strictly linear in m without any periodicity (in fact, $Z_m = (n_2 + n_4 - 1)/2$ and both n_2 and n_4 increase by one unit with m). All four singularities a, b, c, d, which are just 0, C/S, S/C and ∞ in this case, appear at each step and can thus be taken as constant. The four Z_k, $\mathrm{k} = a, b, c, d$, depend on m in a very specific way: $Z_\mathrm{k}(m) = Z_m + (-1)^m \Delta_\mathrm{k}$, where the Δ_k satisfy $\sum_\mathrm{k} \Delta_\mathrm{k} = 0$. Therefore the sum of the numerators on the rhs balances the sum on the lhs in the generic case where all four singularities are at finite distance, which can always be achieved with a homographic transformation. This is the well-known [9, 13] contiguity relation of $\mathrm{P_{VI}}$.

A more complicated case can also be considered. Subtracting (3.18) from (3.22) we get

$$\frac{n_4 + n_2 - 2}{w_{12} - w_{1224}} + \frac{n_2 - n_3}{w_{12} - w_{13}} = \frac{n_1 + n_2 - 2}{w_{12} - \sigma} + \frac{n_2 - n_1}{w_{12} - \nu} + \frac{n_4 - n_3}{w_{12} - \rho}. \tag{3.25}$$

With $w_{m-1} \equiv x_{13}$, $w_m \equiv x_{12}$ to $w_{m+1} \equiv x_{1224}$, we formally get

$$\frac{Z_{m+1/2}}{w_m - w_{m+1}} + \frac{Z_{m-1/2}}{w_m - w_{m-1}} = \frac{Z_a}{w_m - a_m} + \frac{Z_b}{w_m - b_m} + \frac{Z_c}{w_m - c_m}. \tag{3.26}$$

The trajectory is a complicated one in the weight lattice of $\mathrm{D}_4^{(1)}$. All four singularities must appear on the rhs somewhere on the trajectory. One can choose for instance $a_m \equiv \sigma$ and $b_m \equiv \nu$ present at each step while for c_m, 0 and ∞ alternate. We first introduce $Z_{m+1/2} = z_m + z_{m+1}$ with $z_m = pm + q + r\,\mathrm{j}^m + s\,\mathrm{j}^{2m}$. Moreover we have $Z_a = z_m + k(-1)^m$, $Z_b = z_m - k(-1)^m$, $Z_c = z_{m+1} + z_{m-1}$. Note that $Z_a + Z_b + Z_c = Z_{m-1/2} + Z_{m+1/2}$, but the term involving Z_c does not appear on every other step when $c_m = \infty$.

To sum it up, our equations (3.17) (of which (3.8) and (3.16) are instances), (3.20) (of which (3.19) and (3.21) are instances) and (3.26) (of which (3.25) is an instance), are indeed new as stressed again here. Mappings (3.17), (3.20) and (3.26) are distinct from (3.24): the evolution described by (3.24) is in a straight line. In any instance involving three consecutive points, the central point is indeed the midpoint of the two extreme ones. In all cases, the distance between consecutive points is the square root of two in the appropriate units (in the representation we use throughout the paper, the second one described in section 3). In the case of (3.17), an instance relates three points forming an equilateral triangle. These triangles wind up, in the four-dimensional $\mathrm{D}_4^{(1)}$ space in a way that, lacking four-dimensional vision, we cannot have a clear visual representation. But one can check that the successive displacements have a complicated, fully four-dimensional, character. In the case of (3.20), an instance relates three points forming a right isosceles triangle. These triangles, as we have said just after equation (3.20), all fit within a two-dimensional plane and form a 'staircase' whose overall direction makes a $\pi/4$ angle with the evolution described by (3.24), each 'step' of the staircase being either parallel or perpendicular to the direction of (3.24). In the case of (3.26), an instance relates three points forming an isosceles triangle with angles $\frac{2\pi}{3}$ and twice $\pi/6$. Again, these triangles wind up, in the four-dimensional $\mathrm{D}_4^{(1)}$ space in a way for which we do not have a visual representation. In these three cases, the basic element is *not* made of three points in straight line and, for this reason, is not reducible to that of (3.24).

4. Conclusion

In this paper, we have obtained the contiguity relations of the solutions of the continuous $\mathrm{P_{VI}}$ which introduce new discrete Painlevé equations. The latter were obtained by considering different trajectories consisting in infinite repetitions of a non-closed pattern on the weight lattice of the affine Weyl group $\mathrm{D}_4^{(1)}$ which describes the transformations of $\mathrm{P_{VI}}$. The elementary building block was the Miura transformation which relates to the solutions of $\mathrm{P_{VI}}$ for three different values of the parameters which correspond to the vertices of an equilateral triangle in the aforementioned space.

While all the contiguity relations derived above concerned the Painlevé VI equation one may wonder whether analogous relations do exist for its 'modified' form, namely the Bureau–Ablowitz–Fokas equation. It turns out that this is indeed possible and does not present particular difficulties. Using (3.4) one can write an equation for ϕ_1 at point ξ_1 relating it to the analogous variables at nearby equivalent points. In particular, we consider the points ζ at

$$\begin{bmatrix}-1/2\\-1/2\\-1/2\\-1/2\end{bmatrix}$$

and ζ' at

$$\begin{bmatrix}-3/2\\1/2\\1/2\\1/2\end{bmatrix}.$$

We define $\varphi_{2m} = \phi_1 - (n_2 + n_3 + n_4)/3$ and similarly $\varphi_{2m\pm1}$ at the two points ζ, ζ'. Then we find

$$\begin{aligned}(\varphi_{2m-1} + \varphi_{2m})(\varphi_{2m} + \varphi_{2m+1})\\ = \frac{C^2}{S^2}\frac{\left(\varphi_{2m} + \frac{2}{3}(2n_2 - n_3 - n_4)\right)\left(\varphi_{2m} + \frac{2}{3}(2n_3 - n_4 - n_2)\right)\left(\varphi_{2m} + \frac{2}{3}(2n_4 - n_2 - n_3)\right)}{\varphi_{2m} + \frac{4}{3}(n_2 + n_3 + n_4)}.\end{aligned} \tag{4.1}$$

The three quantities added to φ in the numerator of the rhs of (4.1) are independent of m (which is *not* the case for the denominator). Complementing this equation by its 'odd' instance (in which case it turns out that the 'prefactor' C^2/S^2 becomes C^2) we find the equation

$$\begin{aligned}(\varphi_{2m-1} + \varphi_{2m})(\varphi_{2m} + \varphi_{2m+1}) &= \frac{1}{T}\frac{(\varphi_{2m} - z_a)(\varphi_{2m} - z_b)(\varphi_{2m} - z_c)}{(\varphi_{2m} - z_{2m})},\\ (\varphi_{2m} + \varphi_{2m+1})(\varphi_{2m+1} + \varphi_{2m+2}) &= \frac{1}{1-T}\frac{(\varphi_{2m+1} + z_a)(\varphi_{2m+1} + z_b)(\varphi_{2m+1} + z_c)}{(\varphi_{2m} - z_{2m+1})},\end{aligned} \tag{4.2}$$

where, as we remarked above, z_a, z_b, z_c are combinations of n_i which are constant along the motion and z_m is linear in m. The sum of the inverses of the prefactors is unity (remember S and C are hyperbolic sine and cosine, respectively) and the canonical choice (3.3) was indeed $T = S^2/C^2$. This is the contiguity relation for the solutions of the Bureau–Ablowitz–Fokas equation.

The present study does not exhaust all the second-order discrete Painlevé equations one can obtain as contiguity relations of the solutions of the sixth Painlevé equation. As we explained in [14] an infinite number of them can indeed be constructed. Any trajectory consisting in infinite repetitions of a non-closed pattern on the lattice of $\mathrm{D}_4^{(1)}$ would generate such a discrete Painlevé equation.

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